HD-THEP-97-16, hep-lat/9705003

Lattice-continuum relations for 3d SU(N)+Higgs theories

M. Laine [1]

Institut für Theoretische Physik, Philosophenweg 16, D-69120 Heidelberg, Germany

A. Rajantie [2]

Department of Physics, P.O. Box 9, FIN-00014 University of Helsinki, Finland

Abstract

3d lattice studies have recently attracted a lot of attention, especially in connection with finite temperature field theories. One ingredient in these studies is a perturbative computation of the 2-loop lattice counterterms, which are exact in the continuum limit. We extend previous such results to SU(N) gauge theories with Higgs fields in the fundamental and adjoint representations. The fundamental SU(3)×SU(2) case might be relevant for the electroweak phase transition in the MSSM, and the adjoint case for the GUT phase transition and for QCD in the high temperature phase. We also revisit the standard SU(2)×U(1) and U(1) theories.

1 Introduction

Finite temperature gauge theories exhibit many interesting and important phenomena, such as the EW (electroweak) and QCD phase transitions. Unfortunately these are not easy questions to answer, due in part to the infrared problem at finite temperature. However, recently a successful approach has been developed at least for the thermodynamical aspects of the EW phase transition (for a review, see [1]). This approach consists of a perturbative dimensional reduction into a 3d effective theory [2–7] and of non-perturbative

[1] m.laine@thphys.uni-heidelberg.de
[2] arttu.rajantie@helsinki.fi

lattice simulations in that theory [8–13]. Other cases where similar 3d simulations are needed are QCD in the high temperature phase [14] and the Ginzburg-Landau model of superconductivity [15,16].

One ingredient in the non-perturbative 3d studies is the relation of the lattice and continuum regularization schemes [17,18]. This problem is analogous to the determination of $\Lambda_{\overline{\rm MS}}/\Lambda_{\rm latt}$ in 4d [19]. In this paper, we extend previous 3d results to theories relevant for extensions of the Standard Model.

To be more specific, consider that one has to convert the results of lattice simulations to continuum observables. In 4d this is usually done by measuring some experimentally known observables and using them to fix the parameters. However, in 3d this is not possible since no such observables exist. Thus one has to use lattice perturbation theory [20] to explicitly renormalize the theory and find the relationship to the $\overline{\rm MS}$ scheme observables in this way. For super-renormalizable theories the result will be exact in the continuum limit at a finite order in the loop expansion; in 3d one needs to go to 2 loops, and only the mass parameter gets renormalized.

The task of relating lattice and continuum observables for 3d SU(2)+Higgs theories was previously discussed in [17,18]. In [18] the calculations were shown in some detail and results were given for SU(2) and U(1) theories with Higgs fields in adjoint and fundamental representations. Here we go to SU(N) gauge fields with both types of scalar fields. 1-loop $\mathcal{O}(a)$-improvement in 3d has been discussed in [21].

It should be noted that for some observables the 2-loop mass counterterms are not needed. Such case is, for instance, the determination of the discontinuity in $\langle\phi^\dagger\phi\rangle/g_3^2$ at the phase transition point (m_c^2) in a theory with only one mass parameter, if one does not need to know the numerical value of m_c^2 [21]. However, in most cases the value of m_c^2 is needed: one needs it in order to make a systematic comparison with 3d perturbation theory (the lattice and perturbative results should be compared for the same parameters), one needs it if there are several mass parameters in the theory as in the SU(3)$\times$SU(2) model discussed in Sec. 5, and one needs it when connection to 4d physics is made, especially in theories sensitive to the value of $T/\Lambda_{\overline{\rm MS}}$ such as QCD in the high-temperature phase [14].

This paper is organized as follows. In Sec. 2 we briefly discuss some general aspects of the problem. In Sec. 3 we calculate the 2-loop lattice counterterms for the SU(N)+adjoint Higgs theory and in Sec. 4 for the SU(N)+fundamental Higgs theory. Some other fundamental theories, namely SU(3)$\times$SU(2) with two Higgses and SU(2)$\times$U(1), U(1) with one Higgs, are discussed in Sec. 5. In Appendix A we discuss the Feynman rules, in App. B some lattice integrals, in App. C the isospin contractions, in App. D the numerical values of the

constants appearing, and in App. E a unified representation for the g^2, g^4-parts of the counterterms.

2 Calculations

In order to find the relationship between the lattice and $\overline{\text{MS}}$ renormalization schemes we have to calculate some physical quantity in both schemes and equate them. As discussed in [18], the simplest choice is the minimum value of the effective potential. Practically this means requiring that the ϕ^2-parts of the effective potentials coincide.

In addition to the ϕ^2-term, one needs to compute the mass-dependent part of the vacuum counterterm. Its definition is that if it is added to the lattice Lagrangian, $\mathcal{L}_{\text{lat}} \to \mathcal{L}_{\text{lat}} + \delta V$, then the mass-dependent part of the vacuum energy density agrees with the $\overline{\text{MS}}$-result. The significance of the vacuum counterterm is that if $\mathcal{L}_{\text{latt}} = m^2\phi^\dagger\phi + \ldots$, then

$$\langle\phi^\dagger\phi\rangle = \frac{dV(\text{min})}{dm^2}. \tag{1}$$

Hence, the relation of $\langle\phi^\dagger\phi\rangle$ measured in continuum and on the lattice *without* adding δV to the Lagrangian is

$$\langle\phi^\dagger\phi\rangle_{\text{cont}} = \langle\phi^\dagger\phi\rangle_{\text{latt}} + \frac{d(\delta V)}{dm^2}. \tag{2}$$

This equation can be used to infer the non-perturbative value of $\langle\phi^\dagger\phi\rangle_{\text{cont}}$ from lattice simulations. Like for the mass counterterm, the 2-loop result is exact in the continuum limit. If one has a phase transition and is only interested in the discontinuity of $\langle\phi^\dagger\phi\rangle$, then the vacuum counterterm does not contribute.

When replacing SU(2) with SU(N), the Feynman rules discussed in [18] get modified and obtain a more complicated form. In most cases this means only replacing ϵ^{ABC} with f^{ABC}, but sometimes more structure is added. The most crucial difference is that the symmetric tensors d^{ABC} are no longer zero. The Feynman rules for the gauge and ghost-gauge vertices can be read from [20], with the replacements $1/4a^2 \to N/12a$ in Eq. (14.39), and $(2/3)(\delta^{AB}\delta^{CD} + \ldots) \to (2/N)(\delta^{AB}\delta^{CD} + \ldots)$ in Eq. (14.44). The sign of the $\overline{c}cAA$-vertex in [20] should also be reversed: the ghost–gluon part of the Lagrangian is

$$S_{\text{ghost}} = \overline{c}^A(p)c^A(p)\widetilde{p}^2 + igf^{ABC}\overline{c}^A(p)A_i^B(q)c^C(r)\,\underset{\sim}{r_i}\,\widetilde{p}_i$$

$$-\frac{1}{24}g^2a^2\Big(f^{ACE}f^{BDE}+f^{ADE}f^{BCE}\Big)\overline{c}^A(p)A_i^C(q)A_i^D(r)c^B(s)\widetilde{s}_i\widetilde{p}_i, \qquad (3)$$

where momentum conservation and due integrations are implied, and $\widetilde{p}_i = \frac{2}{a}\sin\frac{a}{2}p_i$, $\underset{\sim}{r_i} = \cos\frac{a}{2}r_i$. The other vertices are given in Appendix A.

We will perform the calculations in the R_ξ gauge with $\xi = 1$ in order to make the gauge propagator simple – the result is gauge fixing independent. As mentioned, we only need to go to 2-loop level to obtain the exact result. In [18] the results for most of the necessary integrals were given. In the present case, different components of the fields get different masses in the symmetry breakdown and we have to consider the corresponding integrals. However, it is easy to express all these new integrals using the ones defined in [18]; the most important cases are for completeness given in Appendix B.

3 Adjoint Higgs

Let us consider first a system with an SU(N) gauge field and a Higgs scalar field in the adjoint representation of the gauge group. The Lagrangian on the lattice is

$$\begin{aligned}\mathcal{L}_{\rm latt} = &\frac{1}{a^4g^2}\sum_{i,j}\mathrm{Tr}\left[\mathbf{1}-P_{ij}(x)\right]\\ &+\frac{2}{a^2}\sum_i\Big[\mathrm{Tr}\Phi(x)^2-\mathrm{Tr}\Phi(x)U_i(x)\Phi(x+i)U_i^{-1}(x)\Big]\\ &+m^2\mathrm{Tr}\Phi^2+\lambda_1(\mathrm{Tr}\Phi^2)^2+\lambda_2\mathrm{Tr}\Phi^4, \end{aligned} \qquad (4)$$

where $U_i(x) = \exp[iagA_i(x)]$, $A_i = A_i^AT^A$, $x+i \equiv x+a\vec{e}_i$,

$$P_{ij}(x) = U_i(x)U_j(x+i)U_i^{-1}(x+j)U_j^{-1}(x), \qquad (5)$$

and a is the lattice constant.

When $a \to 0$, this becomes the usual continuum Lagrangian

$$\mathcal{L}_{\rm cont} = \frac{1}{2}\mathrm{Tr}F_{ij}F_{ij}+\mathrm{Tr}D_i\Phi D_i\Phi+m^2\mathrm{Tr}\Phi^2+\lambda_1(\mathrm{Tr}\Phi^2)^2+\lambda_2\mathrm{Tr}\Phi^4, \qquad (6)$$

where $F_{ij} = \partial_iA_j-\partial_jA_i+ig[A_i,A_j]$ and $D_i\Phi = \partial_i\Phi+ig[A_i,\Phi]$.

We can write $\Phi = \sum_{A=1}^{N^2-1}\Phi^AT^A$, where the generators T^A of the Lie algebra of the group SU(N) are chosen to be Hermitian $N\times N$ matrices and are

normalized as $\mathrm{Tr} T^A T^B = \frac{1}{2}\delta^{AB}$. If $\hat{\phi}$ is a real N-dimensional vector, then

$$T^{N^2-1} = \frac{1}{\sqrt{2N(N-1)}} \left(\mathbf{1} - N \frac{\hat{\phi}\hat{\phi}^\dagger}{\hat{\phi}^\dagger \hat{\phi}} \right) \tag{7}$$

is Hermitian, traceless and properly normalized and can therefore be chosen to be one of the generators.

We wish to relate the lattice observables to the $\overline{\mathrm{MS}}$ scheme ones using the method given in [17,18]. In order to calculate the effective potential, we break the symmetry by shifting the Higgs field $\Phi \to \Phi + vT^{N^2-1}$. The effective potential in the adjoint case depends on the direction of the shift. However, we are only interested in the quadratic part of the potential which depends only on the magnitude v and thus any direction can be chosen. The Higgs-gauge cross term is cancelled by introducing an R_ξ gauge fixing term $\mathcal{L}_\xi = F^A F^A / 2\xi a^2$, where

$$F^A(x) = \sum_i \left[A_i^A(x) - A_i^A(x-i) \right] + \xi a g v f^{A,B,N^2-1} \Phi^B(x). \tag{8}$$

If we define the projection operators

$$\begin{aligned}
P_1^{AB} &= \delta^{AB} - P_2^{AB} - P_3^{AB}; \quad P_1^{AA} = N(N-2), \\
P_2^{AB} &= -4 \frac{\hat{\phi}^\dagger T^A \hat{\phi}}{\hat{\phi}^\dagger \hat{\phi}} \frac{\hat{\phi}^\dagger T^B \hat{\phi}}{\hat{\phi}^\dagger \hat{\phi}} + 2 \frac{\hat{\phi}^\dagger T^{\{A} T^{B\}} \hat{\phi}}{\hat{\phi}^\dagger \hat{\phi}}; \quad P_2^{AA} = 2(N-1), \\
P_3^{AB} &= \frac{2N}{N-1} \frac{\hat{\phi}^\dagger T^A \hat{\phi}}{\hat{\phi}^\dagger \hat{\phi}} \frac{\hat{\phi}^\dagger T^B \hat{\phi}}{\hat{\phi}^\dagger \hat{\phi}}; \quad P_3^{AA} = 1,
\end{aligned} \tag{9}$$

we can write the propagators in the broken phase as

$$\begin{aligned}
\langle \Phi^A \Phi^B \rangle &= \left(\frac{P_1^{AB}}{\tilde{k}^2 + m_1^2} + \frac{P_2^{AB}}{\tilde{k}^2 + m_2^2} + \frac{P_3^{AB}}{\tilde{k}^2 + m_3^2} \right), \\
\langle A_i^A A_j^B \rangle &= \delta_{ij} \left(\frac{P_1^{AB}}{\tilde{k}^2} + \frac{P_2^{AB}}{\tilde{k}^2 + M^2} + \frac{P_3^{AB}}{\tilde{k}^2} \right), \quad \langle \overline{c}^A c^B \rangle = -\Big(\ldots \Big),
\end{aligned} \tag{10}$$

where $(\ldots)$ is the same expression as in the gauge propagator and the masses are

$$\begin{aligned}
M^2 &= \frac{N}{2(N-1)} g^2 v^2, \\
m_1^2 &= m^2 + \left(\lambda_1 + \frac{3}{N(N-1)} \lambda_2 \right) v^2,
\end{aligned}$$

$$
\begin{aligned}
m_2^2 &= m^2 + \left(\lambda_1 + \frac{N^2-3N+3}{N(N-1)}\lambda_2\right) v^2 + M^2, \\
m_3^2 &= m^2 + 3\left(\lambda_1 + \frac{N^2-3N+3}{N(N-1)}\lambda_2\right) v^2.
\end{aligned} \tag{11}
$$

It is now straightforward to calculate the renormalization counterterms and to relate them to the ones obtained in the $\overline{\mathrm{MS}}$ scheme. Let $m^2(\mu)$ be the renormalized mass in the $\overline{\mathrm{MS}}$ scheme and $m^2 = m^2(\mu) + \delta m^2(\mu)$ the bare mass. The diagrams needed to calculate δm^2 are the same as in [18]. The Feynman rules for the vertices are given in Appendix A. The calculation of the isospin factors in the broken phase is discussed in Appendix C, and some typical lattice integrals are in Appendix B. For some diagrams, products of four structure constants are needed. Their values are given in Appendix A of [7].

Adding the contributions from the different diagrams together we get

$$
\begin{aligned}
\delta m^2 = &-\left(2Ng^2 + (N^2+1)\lambda_1 + (2N^2-3)\frac{\lambda_2}{N}\right)\frac{\Sigma}{4\pi a} \\
&-\frac{1}{16\pi^2}\left\{\left[2Ng^2\left((N^2+1)\lambda_1 + (2N^2-3)\frac{\lambda_2}{N}\right) - 2(N^2+1)\lambda_1^2\right.\right. \\
&\left. -4(2N^2-3)\lambda_1\frac{\lambda_2}{N} - (N^4-6N^2+18)\frac{\lambda_2^2}{N^2}\right]\left(\ln\frac{6}{a\mu} + \zeta\right) \\
&+2Ng^2\left((N^2+1)\lambda_1 + (2N^2-3)\frac{\lambda_2}{N}\right)\left(\frac{1}{4}\Sigma^2 - \delta\right) \\
&\left. +g^4N^2\left[\frac{5}{8}\Sigma^2 + \left(\frac{1}{2} - \frac{4}{3N^2}\right)\pi\Sigma - 4(\delta+\rho) + 2\kappa_1 - \kappa_4\right]\right\},
\end{aligned} \tag{12}
$$

where the constants ζ, δ, ρ, κ_1, κ_4 and Σ have been defined in [17,18]. Their numerical values are given in Appendix D. The μ dependences of δm^2 and m^2 cancel as they should, which can be seen by comparison with Eq. (48) of [7]. As for SU(2), the finite parts of δ_{ii}/ϵ arising from dimensional regularization in the $\overline{\mathrm{MS}}$ case, cancel between the diagrams (vvv) and (vvs). The 2-loop $1/a$-terms cancel against contributions from the mass counterterms.

The mass-dependent part of the vacuum counterterm is also straightforward to calculate, and it implies that

$$
\begin{aligned}
\left\langle \mathrm{Tr}\Phi^2 \right\rangle_{\mathrm{cont}} = &\left\langle \mathrm{Tr}\Phi^2 \right\rangle_{\mathrm{latt}} - (N^2-1)\frac{\Sigma}{8\pi a} \\
&-\frac{g^2}{16\pi^2}N(N^2-1)\left(\ln\frac{6}{a\mu} + \zeta + \frac{\Sigma^2}{4} - \delta\right).
\end{aligned} \tag{13}
$$

As a concrete example, consider SU(5). Using the numerical values of the constants $\Sigma, \delta, \rho, \zeta, \kappa_1, \kappa_4$ given in Appendix D, one gets from eq. (12),

$$
\begin{aligned}
m^2 = m^2(\mu) &- \left(10g^2 + 26\lambda_1 + \frac{47}{5}\lambda_2\right)\frac{\Sigma}{4\pi a} \\
&- \frac{1}{16\pi^2}\Big\{\Big[10g^2\left(26\lambda_1 + \frac{47}{5}\lambda_2\right) - 52\lambda_1^2 - \frac{188}{5}\lambda_1\lambda_2 - \frac{493}{25}\lambda_2^2\Big] \\
&\times\left(\ln\frac{6}{a\mu} + \zeta\right) + 124.01g^4 + 5.7947g^2\left(26\lambda_1 + \frac{47}{5}\lambda_2\right)\Big\}.
\end{aligned}
\tag{14}
$$

The combination appearing on the last row in eq. (13) is in eq. (D.5).

4 Fundamental Higgs

For the fundamental theory, the lattice Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{\text{lat}} = &\frac{1}{a^4g^2}\sum_{i,j}\mathrm{Tr}\Big[\mathbf{1} - P_{ij}(x)\Big] \\
&+ \frac{2}{a^2}\sum_i\Big[\phi^\dagger(x)\phi(x) - \mathrm{Re}\,\phi^\dagger(x)U_i(x)\phi(x+i)\Big] \\
&+ m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2.
\end{aligned}
\tag{15}
$$

The corresponding continuum theory is

$$
\mathcal{L}_{\text{cont}} = \frac{1}{2}\mathrm{Tr}F_{ij}F_{ij} + (D_i\phi)^\dagger(D_i\phi) + m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \tag{16}
$$

where $F_{ij} = T^AF_{ij}^A$ and $D_i = \partial_i + igT^AA_i^A$.

In contrary to the adjoint case, the effective potential can be calculated in the fundamental case for an arbitrary shift (see, e.g., [22]). We make a shift $\hat{\phi}$ such that

$$
\hat{\phi}^\dagger\hat{\phi} = \frac{v^2}{2}. \tag{17}
$$

For simplicity, we shall assume $\hat{\phi}$ to be real (this assumption enters in the scalar propagators). The gauge fixing term is

$$
F^A = \sum_i[A_i^A(x) - A_i^A(x-i)] + i\xi ag\Big(\phi^\dagger T^A\hat{\phi} - \hat{\phi}^\dagger T^A\phi\Big). \tag{18}
$$

As mentioned, in practical calculations we use $\xi = 1$ to simplify the momentum algebra. The shifted field $\tilde{\phi} = \phi - \hat{\phi}$ is written as $\tilde{\phi}_\alpha = (u_\alpha + i\omega_\alpha)/\sqrt{2}$, $\alpha = 1, \ldots, N$. To get the propagators we define two more projection operators in addition to those in eq. (9),

$$T_{\alpha\beta} = \delta_{\alpha\beta} - \frac{\hat{\phi}_\alpha \hat{\phi}_\beta}{\hat{\phi}^\dagger \hat{\phi}}, \quad L_{\alpha\beta} = \frac{\hat{\phi}_\alpha \hat{\phi}_\beta}{\hat{\phi}^\dagger \hat{\phi}}; \qquad T_{\alpha\alpha} = N - 1, \quad L_{\alpha\alpha} = 1. \tag{19}$$

Then the propagators are

$$\begin{aligned}
&\langle u_\alpha u_\beta \rangle = \frac{T_{\alpha\beta}}{\tilde{k}^2 + m_\omega^2} + \frac{L_{\alpha\beta}}{\tilde{k}^2 + m_u^2}, \quad \langle \omega_\alpha \omega_\beta \rangle = \frac{T_{\alpha\beta}}{\tilde{k}^2 + m_\omega^2} + \frac{L_{\alpha\beta}}{\tilde{k}^2 + \overline{m}_\omega^2}, \\
&\langle A_i^A A_j^B \rangle = \delta_{ij} \Big(\frac{P_1^{AB}}{\tilde{k}^2} + \frac{P_2^{AB}}{\tilde{k}^2 + M^2} + \frac{P_3^{AB}}{\tilde{k}^2 + \overline{M}^2} \Big), \quad \langle \overline{c}^A c^B \rangle = -\Big(\ldots \Big).
\end{aligned} \tag{20}$$

Here $\overline{c}^A, c^B$ are the ghost fields, the expression $(\ldots)$ is the same as for the gauge fields, and the non-vanishing masses are

$$\begin{aligned}
M^2 &= \frac{1}{4} g^2 v^2, \quad \overline{M}^2 = \frac{2(N-1)}{N} M^2, \\
m_\omega^2 &= m^2 + \lambda v^2 + M^2, \quad \overline{m}_\omega^2 = m^2 + \lambda v^2 + \overline{M}^2, \\
m_u^2 &= m^2 + 3\lambda v^2.
\end{aligned} \tag{21}$$

The isospin algebra appearing in the graphs is for the most part the same as in continuum, and can be read from [22]. There are a few additional terms which do not appear in continuum since the vertex is proportional to powers of a^2. Such terms come from the $\overline{c}cAA$-vertex and the two $AAAA$-vertices. The isospin algebra of $\langle \overline{c}cAA \rangle$ can be deduced from the graph (vv) in eq. (B.46) in [22]. The algebra related to the $AAAA$-vertex in eq. (A.4) is

$$\Big(\delta^{AB}\delta^{CD} + \delta^{AC}\delta^{BD} + \delta^{AD}\delta^{BC} \Big) \frac{\phi^\dagger T^A T^B T^C T^D \phi}{\phi^\dagger \phi} = \frac{(N^2-1)(2N^2-3)}{4N^2},$$

where we used that the masses in the propagators can be put to zero here. The algebra related to the symmetric a^4-part of the four-gluon vertex coming from the plaquette is

$$\begin{aligned}
&\langle A^A A^B \rangle \langle A^C A^D \rangle \Big[\Big(\frac{2}{N} \delta^{AB}\delta^{CD} + d^{ABE} d^{CDE} \Big) + (B \leftrightarrow C) + (B \leftrightarrow D) \Big] \\
&\to \frac{4}{N}(2N^2 - 3)\delta^{CD} \langle A^C A^D \rangle_{M^2-\text{part}} = \frac{4(N^2-1)(2N^2-3)}{N^2} g^2 \hat{\phi}^\dagger \hat{\phi}.
\end{aligned} \tag{22}$$

Here we used eq. (B.9) according to which the result of the momentum integral is symmetric and quadratic in the masses in the continuum limit.

After the momentum integrals, one can verify the cancellation of $1/a$-terms against 1-loop counterterm contributions. The result left is

$$\begin{aligned}
m^2 = m^2(\mu) &- \left(\frac{N^2-1}{N}g^2 + 2(N+1)\lambda\right)\frac{\Sigma}{4\pi a} \\
&-\frac{1}{16\pi^2}(N^2-1)\Big\{\Big[g^4\frac{4N^2-N+3}{4N^2} + 2\lambda g^2\frac{N+1}{N} - 4\lambda^2\frac{1}{N-1}\Big] \\
&\times\left(\ln\frac{6}{a\mu}+\zeta\right) + 2\lambda g^2\frac{N+1}{N}\left(\frac{1}{4}\Sigma^2-\delta\right) \\
&+g^4\frac{1}{4N^2}\Big[\frac{4N^2-1}{4}\Sigma^2 + \frac{3N^2-8}{3}\pi\Sigma + N^2+1 \\
&-4N(N+1)\rho - 2(3N^2-1)\delta + 2N^2(2\kappa_1-\kappa_4)\Big]\Big\}.
\end{aligned} \tag{23}$$

In fact, the 1-loop terms proportional to g^2 and the 2-loop terms proportional to g^4 in eqs. (12), (23) can be written in a unified way, and for completeness this formula is given in Appendix E.

The relation of $\langle\phi^\dagger\phi\rangle$ measured in continuum and on the lattice is

$$\begin{aligned}
\langle\phi^\dagger\phi\rangle_{\rm cont} = \langle\phi^\dagger\phi\rangle_{\rm latt} &- \frac{N\Sigma}{4\pi a} \\
&-\frac{1}{(4\pi)^2}g^2(N^2-1)\left(\ln\frac{6}{a\mu}+\zeta+\frac{1}{4}\Sigma^2-\delta\right).
\end{aligned} \tag{24}$$

Corrections of order a to the discontinuity of $\langle\phi^\dagger\phi\rangle$ are discussed in [21].

5 Other fundamental Higgs theories

5.1 SU(N)×SU(2)

It has been argued in [6,22] that an SU(3)×SU(2) gauge theory with two scalar fields might be relevant for the electroweak phase transition in MSSM. Let us here consider for generality SU(N)×SU(2). The continuum Lagrangian is

$$\begin{aligned}
\mathcal{L}_{\rm cont} = \frac{1}{4}F^a_{ij}F^a_{ij} + \frac{1}{4}G^A_{ij}G^A_{ij} + (D^w_i H)^\dagger(D^w_i H) + m_H^2 H^\dagger H + \lambda_H(H^\dagger H)^2 \\
+ (D^s_i U)^\dagger(D^s_i U) + m_U^2 U^\dagger U + \lambda_U(U^\dagger U)^2 + \gamma H^\dagger H U^\dagger U.
\end{aligned} \tag{25}$$

Here $D_i^w = \partial_i + ig_W t^a A_i^a$ and $D_i^s = \partial_i + ig_S T^A C_i^A$ are the SU(2) and SU(N) covariant derivatives ($t^a = \sigma^a/2$ where σ^a are the Pauli matrices), g_W and g_S are the corresponding gauge couplings, H is the Higgs doublet and U is the right-handed stop field. Since the two scalar fields are coupled only through the simple scalar vertex proportional to γ, discretization follows immediately from Sec. 4.

The results for the counterterms can also mostly be read from Sec. 4. The reason is that when one makes a shift in H, say, then in the SU(N)-sector one only needs to replace $m_U^2 \to m_U^2 + \gamma\hat{H}^\dagger\hat{H}$ in the masses in eq. (21). Thus the result can be deduced from the mass-dependent part of the vacuum counterterm. The only new graph is the scalar figure-8 graph proportional to γ^2.

The results for the bare lattice mass parameters are

$$
\begin{aligned}
m_U^2 = {} & m_U^2(\mu) - \left(\frac{N^2-1}{N}g_S^2 + 2(N+1)\lambda_U + 2\gamma\right)\frac{\Sigma}{4\pi a} \\
& -\frac{1}{16\pi^2}(N^2-1)\Bigg\{\Bigg[g_S^4\frac{4N^2-N+3}{4N^2} + 2\lambda_U g_S^2\frac{N+1}{N} \\
& +\frac{1}{N^2-1}\left(3\gamma g_W^2 - 4(N+1)\lambda_U^2 - 2\gamma^2\right)\Bigg]\left(\ln\frac{6}{a\mu}+\zeta\right) \\
& +\left(2\lambda_U g_S^2\frac{N+1}{N} + \frac{3}{N^2-1}\gamma g_W^2\right)\left(\frac{1}{4}\Sigma^2 - \delta\right) \\
& +g_S^4\frac{1}{4N^2}\Bigg[\frac{4N^2-1}{4}\Sigma^2 + \frac{3N^2-8}{3}\pi\Sigma + N^2 + 1 \\
& -4N(N+1)\rho - 2(3N^2-1)\delta + 2N^2(2\kappa_1-\kappa_4)\Bigg]\Bigg\},
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
m_H^2 = {} & m_H^2(\mu) - \left(\frac{3}{2}g_W^2 + 6\lambda_H + N\gamma\right)\frac{\Sigma}{4\pi a} \\
& -\frac{1}{16\pi^2}\Bigg\{\left[\frac{51}{16}g_W^4 + 9\lambda_H g_W^2 - 12\lambda_H^2 + (N^2-1)\gamma g_S^2 - N\gamma^2\right] \\
& \times\left(\ln\frac{6}{a\mu}+\zeta\right) + \left[9\lambda_H g_W^2 + (N^2-1)\gamma g_S^2\right]\left(\frac{1}{4}\Sigma^2 - \delta\right) \\
& +g_W^4\frac{3}{16}\left[\frac{15}{4}\Sigma^2 + \frac{4}{3}\pi\Sigma + 5 - 24\rho - 22\delta + 8(2\kappa_1-\kappa_4)\right]\Bigg\}.
\end{aligned}
\tag{27}
$$

Using the numerical values in Appendix D, one gets for $N=3$,

$$
\begin{aligned}
m_U^2 = {} & m_U^2(\mu) - \left(\frac{8}{3}g_S^2 + 8\lambda_U + 2\gamma\right)\frac{\Sigma}{4\pi a} \\
& -\frac{1}{16\pi^2}\Bigg\{\left[8g_S^4 + \frac{64}{3}\lambda_U g_S^2 - 16\lambda_U^2 + 3\gamma g_W^2 - 2\gamma^2\right]\left(\ln\frac{6}{a\mu}+\zeta\right)
\end{aligned}
$$

$$+19.633g_S^4+12.362\lambda_U g_S^2+1.7384\gamma g_W^2\Big\}, \tag{28}$$

$$\begin{aligned} m_H^2 = & m_H^2(\mu) - \left(\frac{3}{2}g_W^2+6\lambda_H+3\gamma\right)\frac{\Sigma}{4\pi a} \\ & -\frac{1}{16\pi^2}\Big\{\left[\frac{51}{16}g_W^4+9\lambda_H g_W^2-12\lambda_H^2+8\gamma g_S^2-3\gamma^2\right]\left(\ln\frac{6}{a\mu}+\zeta\right) \\ & +4.9941g_W^4+5.2153\lambda_H g_W^2+4.6358\gamma g_S^2\Big\}. \end{aligned} \tag{29}$$

The vacuum counterterms do not get any corrections from γ, so that

$$\begin{aligned} \langle U^\dagger U\rangle_{\text{cont}} = & \langle U^\dagger U\rangle_{\text{latt}} - \frac{N\Sigma}{4\pi a} - \frac{1}{(4\pi)^2}(N^2-1)g_S^2\left(\ln\frac{6}{a\mu}+\zeta+\frac{1}{4}\Sigma^2-\delta\right), \\ \langle H^\dagger H\rangle_{\text{cont}} = & \langle H^\dagger H\rangle_{\text{latt}} - \frac{\Sigma}{2\pi a} - \frac{1}{(4\pi)^2}3g_W^2\left(\ln\frac{6}{a\mu}+\zeta+\frac{1}{4}\Sigma^2-\delta\right). \end{aligned} \tag{30}$$

The numerical value of the constant $\zeta+\Sigma^2/4-\delta$ is in eq. (D.5).

5.2 SU(2)×U(1)

For the g'^2-part of standard electroweak SU(2)×U(1) theory, only the numerical values of the lattice counterterms have been previously given in literature [9]. We give here the exact result in terms of the lattice constants $\Sigma, \zeta, \delta, \rho, \kappa_1, \kappa_4$.

The SU(2)×U(1) lattice Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{latt}} = & \frac{1}{a^4g^2}\sum_{i,j}\text{Tr}\Big[\mathbf{1}-P_{ij}(x)\Big] + \frac{2\gamma^2}{a^4g'^2}\sum_{i,j}[1-p_{ij}^{1/\gamma}(x)] \\ & +\frac{1}{a^2}\sum_i\Big[\text{Tr}\Phi^\dagger(x)\Phi(x)-\text{Tr}\Phi^\dagger(x)U_i(x)\Phi(x+i)e^{-i\alpha_i(x)\sigma_3}\Big] \\ & +m^2\frac{1}{2}\text{Tr}\Phi^\dagger(x)\Phi(x)+\lambda\left[\frac{1}{2}\text{Tr}\Phi^\dagger(x)\Phi(x)\right]^2. \end{aligned} \tag{31}$$

Here p_{ij} is the U(1) plaquette with the U(1) link $u_i(x)=\exp[i\alpha_i(x)]$, where $\alpha_i(x)=ag'B_i(x)/2$. Any positive number γ in eq. (31) gives the same naive continuum limit. The Higgs field has been written as a matrix $\Phi=(\tilde{\phi}\,\phi)$, where ϕ is the standard Higgs doublet and $\tilde{\phi}=i\sigma^2\phi^*$. The continuum Lagrangian corresponding to eq. (31) is

$$L=\frac{1}{4}F_{ij}^aF_{ij}^a+\frac{1}{4}B_{ij}B_{ij}+(D_i\phi)^\dagger D_i\phi+m^2\phi^\dagger\phi+\lambda(\phi^\dagger\phi)^2, \tag{32}$$

where $D_i = \partial_i + igA_i + ig'B_i/2$ and $B_{ij} = \partial_i B_j - \partial_j B_i$.

The calculation proceeds as before. The bare lattice mass parameter is

$$
\begin{aligned}
m^2 = {} & m^2(\mu) - \frac{\Sigma}{8\pi a}(3g^2 + g'^2 + 12\lambda) \\
& - \frac{1}{16\pi^2}\Big[\Big(\frac{51}{16}g^4 - \frac{9}{8}g^2g'^2 - \frac{5}{16}g'^4 + 9\lambda g^2 - 12\lambda^2 + 3\lambda g'^2\Big)\Big(\ln\frac{6}{a\mu} + \zeta\Big) \\
& + \frac{3}{2}g^4\Big(\frac{15}{32}\Sigma^2 + \frac{5}{8} + \frac{\pi\Sigma}{6} - \frac{11}{4}\delta - 3\rho + 2\kappa_1 - \kappa_4\Big) \\
& + \frac{3}{8}g^2g'^2\Big(\frac{1}{4}\Sigma^2 - 1 - 2\delta\Big) + \frac{1}{16}g'^4\Big(\frac{1}{4}\Sigma^2 - 1 - 2\delta - 8\rho + \frac{8\pi}{3}\frac{\Sigma}{\gamma^2}\Big) \\
& + 3\lambda(3g^2 + g'^2)\Big(\frac{\Sigma^2}{4} - \delta\Big)\Big].
\end{aligned} \tag{33}
$$

The numerical value of the constant 2-loop part on the last three rows is

$$
\begin{aligned}
-\frac{1}{16\pi^2}\Big[& 4.9941g^4 - 0.88600g^2g'^2 + \Big(0.00932 + \frac{1.66290}{\gamma^2}\Big)g'^4 \\
& + 5.2153\lambda g^2 + 1.7384\lambda g'^2\Big].
\end{aligned} \tag{34}
$$

The relation of the order parameters is

$$
\begin{aligned}
\langle\phi^\dagger\phi\rangle_{\rm cont} = {} & \langle\phi^\dagger\phi\rangle_{\rm latt} - \frac{\Sigma}{2\pi a} \\
& - \frac{1}{16\pi^2}(3g^2 + g'^2)\Big(\ln\frac{6}{a\mu} + \zeta + \frac{1}{4}\Sigma^2 - \delta\Big).
\end{aligned} \tag{35}
$$

The SU(2)+Higgs theory follows as the limit $g' \to 0$.

5.3 U(1)

For the U(1)+Higgs theory we extend previous results [18] by introducing an extra parameter γ as in Ref. [9] and Sec. 5.2, allowing a simultaneous representation of the compact and non-compact lattice actions.

The U(1)+Higgs theory is defined by

$$
\mathcal{L}_{\rm lat} = \frac{\gamma^2}{2a^4e^2}\sum_{i,j}\Big[1 - P_{ij}^{1/\gamma}(x)\Big]
$$

$$
\begin{aligned}
&+\frac{2}{a^2}\sum_i\Big[\phi^*(x)\phi(x)-\mathrm{Re}\,\phi^*(x)U_i(x)\phi(x+i)\Big]\\
&+m^2\phi^*\phi+\lambda\Big(\phi^*\phi\Big)^2,
\end{aligned}
\tag{36}
$$

where $U_i(x)=\exp[iaeA_i(x)]$. The continuum Lagrangian is

$$
\mathcal{L}_{\rm cont}=\frac{1}{4}F_{ij}F_{ij}+(D_i\phi)^*D_i\phi+m^2\phi^*\phi+\lambda(\phi^*\phi)^2,
\tag{37}
$$

with $D_i=\partial_i+ieA_i$. The bare mass parameter is

$$
\begin{aligned}
m^2 =& m^2(\mu)-(2e^2+4\lambda)\frac{\Sigma}{4\pi a}\\
&-\frac{1}{16\pi^2}\bigg[\Big(-4e^4+8\lambda e^2-8\lambda^2\Big)\Big(\ln\frac{6}{a\mu}+\zeta\Big)\\
&+e^4\Big(\frac{1}{4}\Sigma^2-1-2\delta-4\rho+\frac{8\pi}{3}\frac{\Sigma}{\gamma^2}\Big)+8\lambda e^2\Big(\frac{1}{4}\Sigma^2-\delta\Big)\bigg],
\end{aligned}
\tag{38}
$$

where the numerical value of the last row is

$$
-\frac{1}{16\pi^2}\bigg[\Big(-1.1068+\frac{26.6065}{\gamma^2}\Big)e^4+4.6358\lambda e^2\bigg].
\tag{39}
$$

The relation of the order parameters is

$$
\langle\phi^*\phi\rangle_{\rm cont}=\langle\phi^*\phi\rangle_{\rm latt}-\frac{\Sigma}{4\pi a}-\frac{e^2}{8\pi^2}\Big(\ln\frac{6}{a\mu}+\zeta+\frac{1}{4}\Sigma^2-\delta\Big).
\tag{40}
$$

The non-compact case follows as the limit $\gamma\to\infty$.

6 Conclusions

We have applied the previously developed method of deriving 2-loop lattice-continuum relations to 3d theories relevant for QCD and some extensions of the Standard Model at finite temperature. The main results are the expressions for the bare lattice mass parameters in eqs. (12),(26),(27), with numerical values as given in eqs. (14),(28),(29). When an extrapolation to the continuum limit is made in the lattice simulations, these equations allow to extract the corresponding finite fixed value of the $\overline{\rm MS}$ mass parameter $m^2(\mu)$. Then one can compare lattice simulations with 3d perturbation theory, or if $m^2(\mu)$ has been computed with methods of dimensional reduction from 4d, one can extract the physical temperature to which the simulations correspond.

Acknowledgements

We acknowledge useful discussions with K.Kajantie. M.L is grateful to D.Bödeker for providing his computer program for SU(N) algebra. M.L was partially supported by the University of Helsinki.

A Feynman rules

In this appendix we give the Feynman rules for the interactions of the Higgs field in the symmetric phase. The ones related to gauge and ghost-gauge vertices were discussed in Sec. 2.

In the adjoint case, the vertices can be read from the action

$$
\begin{aligned}
S_\Phi = &\frac{1}{4}\left(\lambda_1 + \frac{\lambda_2}{N}\right)\Phi^A\Phi^A\Phi^B\Phi^B + \frac{1}{8}\lambda_2 d^{ABE}d^{CDE}\Phi^A\Phi^B\Phi^C\Phi^D \\
&+\frac{ig}{2}\delta(p+q+r)\widetilde{(p-q)}_i f^{ABC}\Phi^A(p)\Phi^B(q)A_i^C(r) \\
&+\frac{g^2}{2}\delta(p+q+r+s)\underset{\sim}{(p-q)}_i f^{ACE}f^{BDE}\Phi^A(p)\Phi^B(q)A_i^C(r)A_i^D(s) \\
&-g^2 v f^{A,D,N^2-1}f^{BCD}\overline{c}^A c^B\Phi^C.
\end{aligned}
\tag{A.1}
$$

In the broken phase, the shift generates two new three-point vertices $\Phi\Phi\Phi$ and ΦAA from those in eq. (A.1), but there is also an additional four-gluon vertex

$$
S_A = -\frac{1}{24}a^2 g^4 v^2 f^{AEH}f^{B,E,N^2-1}f^{CFH}f^{D,F,N^2-1}A_i^A A_i^B A_i^C A_i^D. \tag{A.2}
$$

In the fundamental case, correspondingly,

$$
\begin{aligned}
S_\phi = &\lambda(\phi^\dagger\phi)^2 \\
&+ g\delta(-p+q+r)\widetilde{(p+q)}_i\phi^\dagger(p)T^A\phi(q)A_i^A(r) \\
&+ \frac{g^2}{2}\delta(-p+q+r+s)\,\underset{\sim}{(p+q)}_i\,\phi^\dagger(p)T^{\{A}T^{B\}}\phi(q)A_i^A(r)A_i^B(s) \\
&+ g^2\overline{c}^A c^B\left(\hat{\phi}^\dagger T^A T^B\phi + \phi^\dagger T^B T^A\hat{\phi}\right).
\end{aligned}
\tag{A.3}
$$

In the broken phase, the shift generates again the extra vertices $\phi\phi\phi$, ϕAA,

and there is the additional four-gluon vertex

$$S_A = -\frac{1}{12} a^2 g^4 (\hat{\phi}^\dagger T^A T^B T^C T^D \hat{\phi}) A_i^A A_i^B A_i^C A_i^D. \tag{A.4}$$

B Lattice integrals

In [17,18], the lattice integrals were worked out for mass combinations specific to SU(2)+fundamental Higgs. Let us here describe how these generalize to the present case. Let

$$S(k^2, m^2) = \frac{1}{\widetilde{k}^2 + m^2}, \quad V_{ij}^1(k^2, M^2) = \frac{\delta_{ij}}{\widetilde{k}^2 + M^2}, \tag{B.1}$$

$$F_{ijk}(p,q,r) = \delta_{ik}\, \underset{\sim}{q_k} \widetilde{(p_j - r_j)} + \delta_{kj}\, \underset{\sim}{p_j} \widetilde{(r_i - q_i)} + \delta_{ji}\, \underset{\sim}{r_i} \widetilde{(q_k - p_k)}, \tag{B.2}$$

where the superscript in V_{ij}^1 is to remind that we are in the Feynman gauge. Then there are contributions from the following integrals [22]:

$$\begin{aligned}
D^1_{\rm VVV} &= \int dp\, dq\, V_{il}^1(p^2, M_1^2) V_{jm}^1(q^2, M_2^2) V_{kn}^1(r^2, M_3^2) \\
&\qquad\qquad \times F_{ijk}(p,q,r) F_{lmn}(p,q,r) \\
&\to \frac{1}{(4\pi)^2} \sum_{i=1}^{3} \Big[-6 M_i^2 \Big(\ln \frac{6}{a\mu} + \zeta \Big) + \frac{1}{a} \hat{M}_i (8\pi - 15\Sigma) \\
&+ M_i^2 \Big(-\frac{3}{2} + \frac{5}{4}\Sigma^2 - \frac{\pi}{3}\Sigma + 6\delta + 6\rho - 4\kappa_1 + 2\kappa_4 \Big) \Big],
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
D^1_{\rm VVS} &= \int dp\, dq\, V_{il}^1(p^2, M_1^2) V_{jm}^1(q^2, M_2^2) S(r^2, m^2) 4 \delta_{ij}\, \underset{\sim}{r_i}\, \delta_{lm}\, \underset{\sim}{r_l} \\
&\to \frac{1}{(4\pi)^2} \Big[12 \Big(\ln \frac{6}{a\mu} + \zeta \Big) + 2 - \Sigma^2 \Big],
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
D^1_{\rm SSV} &= \int dp\, dq\, S(p^2, m_1^2) S(q^2, m_2^2) V_{ij}^1(r^2, M^2) \widetilde{(p-q)}_i \widetilde{(p-q)}_j \\
&\to \frac{1}{(4\pi)^2} \Big[(M^2 - 2m_1^2 - 2m_2^2) \Big(\ln \frac{6}{a\mu} + \zeta \Big) - \frac{\Sigma}{a} (\hat{m}_1 + \hat{m}_2) \\
&+ \frac{4}{a} \hat{M} \Big(\pi - \frac{3}{2}\Sigma \Big) + 2\delta (m_1^2 + m_2^2) + 4\rho M^2 \Big],
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
D^1_{\rm GGV} &= \int dp\, dq\, S(p^2, m_1^2) S(q^2, m_2^2) V_{ij}^1(r^2, M^2) (-\widetilde{p}_i\, \underset{\sim}{p_j}\, \widetilde{q}_j\, \underset{\sim}{q_i}) \\
&\to \frac{1}{(4\pi)^2} \Big[\frac{1}{2} (M^2 - m_1^2 - m_2^2) \Big(\ln \frac{6}{a\mu} + \zeta \Big) + \frac{1}{a} \hat{M} \Big(\pi - \frac{3}{2}\Sigma \Big) \\
&+ \frac{1}{2} \delta (m_1^2 + m_2^2) + \rho M^2 \Big],
\end{aligned} \tag{B.6}$$

where $r = -p - q$, $\hat{m} \equiv m(1 + \hat{\xi} am)$, $\hat{\xi} \approx 0.152859$ (however, $\hat{\xi}$ cancels in the final result, see [17,18]), and

$$\int dp \equiv \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3}. \tag{B.7}$$

The contributions from the figure-8 graphs follow directly from the SU(2) case in [18], apart from the graph (vv). Let us note that there are two kinds of contributions from this graph, one proportional to $f^{ACE} f^{BDE} \langle A^A A^B \rangle \langle A^C A^D \rangle$ (this is the isospin factor appearing in continuum) and the other to the factor in eq. (22). In the former case the integral appearing is

$$\begin{aligned} D^{1(a)}_{\mathrm{VV}} &= \int dp\, dq\, S(p^2, M_1^2) S(q^2, M_2^2) \Big[12 - \frac{7}{3} a^2 (\tilde{p}^2 + \tilde{q}^2) + \frac{5}{18} a^4 \tilde{p}^2 \tilde{q}^2 \Big] \\ &\to \frac{1}{(4\pi)^2} \sum_{i=1}^{2} \Big[\frac{1}{a} \hat{M}_i \Big(-12\Sigma + \frac{28}{3}\pi \Big) + M_i^2 \Big(\frac{7}{3}\Sigma^2 - \frac{10}{9}\pi\Sigma \Big) \Big], \end{aligned} \tag{B.8}$$

and in the latter case

$$\begin{aligned} D^{1(b)}_{\mathrm{VV}} &= \int dp\, dq\, S(p^2, M_1^2) S(q^2, M_2^2) \Big(-\frac{1}{9} a^4 \tilde{p}^2 \tilde{q}^2 \Big) \\ &\to \frac{1}{(4\pi)^2} (M_1^2 + M_2^2) \frac{4\pi}{9} \Sigma. \end{aligned} \tag{B.9}$$

C Isospin contractions

When calculating the adjoint case mass counterterm in eq. (12) one has to evaluate 2-loop Feynman diagrams with complicated vertices. There are basically two ways to evaluate the isospin factors related to the diagrams. Either one can use the Fiertz identity

$$T^A_{ij} T^A_{kl} = \frac{1}{2} \left(\delta_{il} \delta_{jk} - \frac{\delta_{ij} \delta_{kl}}{N} \right) \tag{C.1}$$

and the projection operators in eq. (9), or one can write the fields in component form and calculate products of the structure constants of the Lie algebra. We give here the necessary products for the latter strategy, which usually leads to simpler algebra.

Let T^A ($A \le N^2 - 1$) be Hermitian traceless $N \times N$ matrices with $\mathrm{Tr} T^A T^B = \frac{1}{2} \delta^{AB}$. They form a representation of the Lie algebra $\mathbf{su}(N)$. Let us define

T^{N^2-1} according to eq. (7). The matrices T^A can be chosen such that the projections P_i^{AB} separate the index set $I = \{1, \ldots, N^2-1\}$ to three subsets I_1, I_2 and I_3. The matrices corresponding to set I_1 form a representation of $\mathbf{su}(N-1)$ and I_3 contains only the index N^2-1.

Since the Feynman rules for the vertices contain factors of the form f^{ABC}, $f^{ABE}f^{CDE}$ and $d^{ABE}d^{CDE}$ it is obvious that one needs to calculate sums like

$$F_{ijk} = \sum_{A\in I_i}\sum_{B\in I_j}\sum_{C\in I_k} f^{ABC}f^{ABC}, \tag{C.2}$$

$$D_{ijk}^{(1)} = \sum_{A\in I_i}\sum_{B\in I_j}\sum_{C\in I_k} d^{ABC}d^{ABC}, \tag{C.3}$$

$$D_{ij}^{(2)} = \sum_{A\in I_i}\sum_{B\in I_j}\sum_{C\in I} d^{AAC}d^{BBC}. \tag{C.4}$$

Using the known results

$$f^{ACD}f^{BCD} = \delta^{AB}N, \quad d^{ACD}d^{BCD} = \delta^{AB}\frac{N^2-4}{N}, \quad d^{AAB} = 0, \tag{C.5}$$

we can deduce the following values for F_{ijk}, $D_{ijk}^{(1)}$ and $D_{ij}^{(2)}$:

$$F_{111} = N(N-1)(N-2), \quad F_{122} = N(N-2), \quad F_{223} = N, \tag{C.6}$$

$$\begin{aligned} D_{111}^{(1)} &= \frac{(N+1)N(N-2)(N-3)}{N-1}, \quad D_{113}^{(1)} = 2\frac{N-2}{N-1}, \\ D_{122}^{(1)} &= N(N-2), \quad D_{223}^{(1)} = \frac{(N-2)^2}{N}, \quad D_{333}^{(1)} = 2\frac{(N-2)^2}{N(N-1)}, \end{aligned} \tag{C.7}$$

$$\begin{aligned} D_{11}^{(2)} &= 2N\frac{(N-2)^2}{N-1}, \quad D_{12}^{(2)} = -2(N-2)^2, \\ D_{13}^{(2)} &= -2\frac{(N-2)^2}{N-1}, \quad D_{22}^{(2)} = \frac{2}{N}(N-1)(N-2)^2, \\ D_{23}^{(2)} &= \frac{2}{N}(N-2)^2, \quad D_{33}^{(2)} = 2\frac{(N-2)^2}{N(N-1)}. \end{aligned} \tag{C.8}$$

The results are symmetric in the permutation of the indices. All other sums are zero.

D Numerical values

Let us use the shorthand $s \equiv \sin$. Then the numerical constants appearing in the 2-loop calculation are as follows:

$$\Sigma = \frac{1}{\pi^2}\int_{-\pi/2}^{\pi/2} d^3x \frac{1}{\sum_i s^2 x_i} = 3.175911535625,$$

$$\delta = \frac{1}{2\pi^4}\int_{-\pi/2}^{\pi/2} d^3x d^3y \frac{\sum_i s^2 x_i s^2(x_i+y_i)}{(\sum_i s^2 x_i)^2 \sum_i s^2(x_i+y_i)\sum_i s^2 y_i} = 1.942130(1),$$

$$\rho = \frac{1}{4\pi^4}\int_{-\pi/2}^{\pi/2} d^3x d^3y \left\{ \frac{\sum_i s^2 x_i s^2(x_i+y_i)}{\sum_i s^2 x_i \sum_i s^2(x_i+y_i)} - \frac{\sum_i s^4 x_i}{(\sum_i s^2 x_i)^2} \right\} \frac{1}{(\sum_i s^2 y_i)^2}$$
$$= -0.313964(1),$$

$$\kappa_1 = \frac{1}{4\pi^4}\int_{-\pi/2}^{\pi/2} d^3x d^3y \frac{\sum_i s^2 x_i s^2(x_i+y_i)}{\sum_i s^2 x_i \sum_i s^2(x_i+y_i)\sum_i s^2 y_i} = 0.958382(1),$$

$$\kappa_4 = \frac{1}{\pi^4}\int_{-\pi/2}^{\pi/2} d^3x d^3y \frac{\sum_i s^2 x_i s^2(x_i+y_i) s^2 y_i}{(\sum_i s^2 x_i)^2 \sum_i s^2(x_i+y_i)\sum_i s^2 y_i} = 1.204295(1). \quad \text{(D.1)}$$

The constant Σ can be expressed in terms of the complete elliptic integral of the first kind [17]. The number in parentheses in the other constants estimates the uncertainty in the last digit.

In [18] two more constants κ_2 and κ_3 were defined. However, they appear in the calculations only as a sum $\kappa_2 + \kappa_3$ and can thus be eliminated using the relation

$$2(\delta + \rho) = \kappa_2 + \kappa_3 + \kappa_4. \quad \text{(D.2)}$$

This follows from the trigonometric identity

$$\sin^4 x + \sin^4 y + \sin^4(x+y) + 4\sin^2 x \sin^2 y \sin^2(x+y)$$
$$-2\left(\sin^2 x \sin^2 y + \sin^2 x \sin^2(x+y) + \sin^2 y \sin^2(x+y)\right) = 0. \quad \text{(D.3)}$$

In addition, there is the constant ζ for which one to our knowledge cannot write a direct expression in momentum space:

$$\zeta = \lim_{z\to 0}\left[\frac{1}{4\pi^4}\int_{-\pi/2}^{\pi/2} \frac{d^3x d^3y}{(\sum_i s^2 x_i + z^2)\sum_i s^2(x_i+y_i)\sum_i s^2 y_i} - \ln\frac{3}{z} - \frac{1}{2}\right]$$
$$= 0.08849(1). \quad \text{(D.4)}$$

For the numerical evaluation this integral can be reduced to a four-dimensional one [17]. Note also that for the combination appearing in the relation of lattice

and continuum order parameters in eqs. (13),(24),(30),(35),(40), one gets

$$\zeta + \frac{1}{4}\Sigma^2 - \delta = 0.66796(1). \tag{D.5}$$

Finally, let us point out that the results above follow from a straightforward numerical integration. If needed, more precise values could be obtained using the position space techniques developed in [23]. We these techniques one can, for instance, write a closed expression for ζ. Noting from eq. (11) in [24] that replacing the mass $am = 2z$ in eq. (D.4) with an external momentum ap causes the numerical factor 1/2 to be replaced with 3/2, one gets

$$\zeta = -\frac{3}{2} - \ln 6 + \sum_x \Big[(4\pi)^2 G(x)^3 + \frac{1}{2}H(x)\Big]. \tag{D.6}$$

Here (we have put $a = 1$ as in [23])

$$G(x) = \int_{-\pi}^{\pi} \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot x}}{\widetilde{p}^2}, \quad H(x) = \int_{-\pi}^{\pi} \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \ln \widetilde{p}^2. \tag{D.7}$$

The functions $G(x), H(x)$ are related by $H(x) = 2\sum_i \Big[G(x) - G(x-i)\Big]/(\sum_i x_i)$ for $\sum_i x_i \neq 0$, and have at $x = 0$ the values

$$G(0) = \frac{\Sigma}{4\pi}, \quad H(0) = 1.67338930297(1). \tag{D.8}$$

The sum in eq. (D.6) is convergent since the leading asymptotic behaviours

$$G(x) \sim \frac{1}{4\pi|x|}, \quad H(x) \sim -\frac{1}{2\pi|x|^3} \tag{D.9}$$

cancel in eq. (D.6). To evaluate the sum one needs an efficient method of calculating $G(x)$ very precisely [23].

E A unified representation for the gauge part of the counterterms

Let $F^A_{BC} = -if^{ABC}$ be the generators of the adjoint representation, and G^A those of a general representation. We define T such that for a real (adjoint) representation,

$$\mathrm{Tr} F^A F^B = T\delta^{AB}, \tag{E.1}$$

and for a complex (fundamental) representation,

$$2\mathrm{Tr}T^AT^B = T\delta^{AB}. \tag{E.2}$$

The quadratic Casimir C_2 is defined by

$$(G^AG^A)_{ab} = C_2\delta_{ab}. \tag{E.3}$$

In addition, the number of real scalar field components is denoted by d. Then for the fundamental representation, $T = 1$, $C_2 = (N^2-1)/2N$ and $d = 2N$, and for the adjoint representation, $T = C_2 = N$, $d = N^2-1$.

In the limit that the scalar self coupling constants vanish, one can then write the bare mass parameter in a unified way:

$$\begin{aligned}\delta m^2 = & -2g^2C_2\frac{\Sigma}{4\pi a} \\ & -\frac{g^4}{16\pi^2}C_2\Big[\left(\frac{7}{2}N-3C_2-\frac{1}{2}T\right)\left(\ln\frac{6}{a\mu}+\zeta\right) \\ & +\left(\frac{3}{8}N+\frac{1}{4}C_2\right)\Sigma^2+\left(\frac{N}{2}-\frac{4}{3N}\right)\pi\Sigma+N-C_2 \\ & -2\left(N+T\right)\rho-2\left(N+C_2\right)\delta+N\left(2\kappa_1-\kappa_4\right)\Big].\end{aligned} \tag{E.4}$$

The mass-dependent part of the vacuum counterterm is

$$\delta V = -m^2d\Big[\frac{\Sigma}{8\pi a}+\frac{g^2}{(4\pi)^2}C_2\Big(\ln\frac{6}{a\mu}+\zeta+\frac{\Sigma^2}{4}-\delta\Big)\Big]. \tag{E.5}$$

References

[1] V.A. Rubakov and M.E. Shaposhnikov, Usp. Fiz. Nauk 166 (1996) 493 [hep-ph/9603208].

[2] P. Ginsparg, Nucl. Phys. B 170 (1980) 388; T. Appelquist and R. Pisarski, Phys. Rev. D 23 (1981) 2305; S. Nadkarni, Phys. Rev. D 27 (1983) 917.

[3] K. Farakos, K. Kajantie, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B 425 (1994) 67; K. Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B 458 (1996) 90 [hep-ph/9508379].

[4] A. Jakovác, K. Kajantie and A. Patkós, Phys. Rev. D 49 (1994) 6810; A. Jakovác and A. Patkós, Phys. Lett. B 334 (1994) 391; hep-ph/9609364.

[5] E. Braaten and A. Nieto, Phys. Rev. D 51 (1995) 6990; Phys. Rev. D 53 (1996) 3421.

[6] J.M. Cline and K. Kainulainen, Nucl. Phys. B 482 (1996) 73 [hep-ph/9605235]; McGill/97-7 [hep-ph/9705201]; M. Losada, RU-96-25 [hep-ph/9605266]; G.R. Farrar and M. Losada, RU-96-26 [hep-ph/9612346]; M. Laine, Nucl. Phys. B 481 (1996) 43 [hep-ph/9605283].

[7] A. Rajantie, hep-ph/9702255.

[8] K. Kajantie, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B 407 (1993) 356; K. Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B 466 (1996) 189; Phys. Rev. Lett. 77 (1996) 2887.

[9] K. Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B, in press [hep-lat/9612006].

[10] E.-M. Ilgenfritz, J. Kripfganz, H. Perlt and A. Schiller, Phys. Lett. B 356 (1995) 561; M. Gürtler, E.-M. Ilgenfritz, J. Kripfganz, H. Perlt and A. Schiller, hep-lat/9512022; Nucl. Phys. B 483 (1997) 383; M. Gürtler, E.-M. Ilgenfritz and A. Schiller, UL-NTZ 08/97 [hep-lat/9702020]; UL-NTZ 10/97 [hep-lat/9704013].

[11] F. Karsch, T. Neuhaus, A. Patkós and J. Rank, Nucl. Phys. B 474 (1996) 217.

[12] O. Philipsen, M. Teper and H. Wittig, Nucl. Phys. B 469 (1996) 445; A. Hart, O. Philipsen, J.D. Stack and M. Teper, Phys. Lett. B 396 (1997) 217.

[13] G.D. Moore and N. Turok, DAMTP-96-77 [hep-ph/9608350].

[14] K. Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, CERN-TH/97-81 [hep-ph/9704416].

[15] M. Karjalainen and J. Peisa, Z. Phys. C, in press [hep-lat/9607023]; K. Kajantie, M. Karjalainen, M. Laine and J. Peisa, CERN-TH/97-62 [cond-mat/9704056].

[16] P. Dimopoulos, K. Farakos, G. Koutsoumbas, NTUA-63-97 [hep-lat/9703004].

[17] K. Farakos, K. Kajantie, K. Rummukainen, and M. Shaposhnikov, Nucl. Phys. B 442 (1995) 317 [hep-lat/9412091].

[18] M. Laine, Nucl. Phys. B 451 (1995) 484 [hep-lat/9504001].

[19] A. Hasenfratz and P. Hasenfratz, Phys. Lett. B 93 (1980) 165; R. Dashen and D.J. Gross, Phys. Rev. D 23 (1981) 2340.

[20] H.J. Rothe, "Lattice Gauge Theories: An Introduction" (World Scientific, Singapore, 1991).

[21] G.D. Moore, PUPT-1649 [hep-lat/9610013].

[22] D. Bödeker, P. John, M. Laine and M.G. Schmidt, Nucl. Phys. B, in press [hep-ph/9612364].

[23] M. Lüscher and P. Weisz, Nucl. Phys. B 445 (1995) 429.

[24] A.K. Rajantie, Nucl. Phys. B 480 (1996) 729 [hep-ph/9606216].